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FREE ACTIONS OF FINITE GROUPS ON 3-DIMENSIONAL NILMANIFOLDS WITH HOMOTOPICALLY TRIVIAL TRANSLATIONS

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Abstract. We show that if a finite group G acts freely with homotopically trivial translations on a 3-dimensional nilmanifold $\mathcal{N}_p$ with the first homology $\mathbb{Z}^2 \oplus \mathbb{Z}_p$, then either G is cyclic or there exist finite nonabelian groups acting freely on $\mathcal{N}_p$ which yield orbit manifolds homeomorphic to $\mathcal{N}/\pi_3$ or $\mathcal{N}/\pi_4$.

1. Introduction

Let $\widetilde{X}$ be a connected, simply connected space with a properly discontinuous action of a discrete group Γ so that it acts as a covering transformations. Let G be a group acting on the manifold $M = \Gamma\backslash\widetilde{X}$. Let $\widetilde{G}$ be the group of liftings of G to the universal covering so that $\widetilde{G} \subset \text{Homeo}(\widetilde{X})$. This fits the short exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1.$$

Let $\mathcal{N}$ be the 3–dimensional Heisenberg group; i.e. $\mathcal{N}$ consists of all 3×3 real upper triangular matrices with diagonal entries 1. Thus $\mathcal{N}$ is a simply connected, 2-step nilpotent Lie group, and it fits an exact sequence

$$1 \to \mathbb{R} \to \mathcal{N} \to \mathbb{R}^2 \to 1$$

where $\mathbb{R} = \mathcal{Z}(\mathcal{N})$, the center of $\mathcal{N}$. Hence $\mathcal{N}$ has the structure of a line bundle over $\mathbb{R}^2$. We take a left invariant metric coming from the

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orthonormal basis

$$\left\{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right\}$$

for the Lie algebra of $\mathcal{N}$. This is, what is called, the Nil-geometry and its isometry group is $\mathrm{Isom}(\mathcal{N}) = \mathcal{N} \rtimes O(2)$ [13]. All isometries of $\mathcal{N}$ preserve orientation and the bundle structure.

We say that a closed 3-dimensional manifold M has a *Nil-geometry* if there is a subgroup π of $\mathrm{Isom}(\mathcal{N})$ so that π acts properly discontinuously and freely with quotient $M = \mathcal{N}/\pi$. The simplest such a manifold is the quotient of $\mathcal{N}$ by the lattice consisting of integral matrices. For each integer $p > 0$, let

$$\Gamma_p = \left\{ \begin{bmatrix} 1 & l & \frac{n}{p} \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix} \middle| \; l, m, n \in \mathbb{Z} \right\}.$$

Then Γ_1 is the discrete subgroup of $\mathcal{N}$ consisting of all matrices with integer entries and Γ_p is a lattice of $\mathcal{N}$ containing Γ_1 with index p. Clearly

$$\mathrm{H}_1(\mathcal{N}/\Gamma_p; \mathbb{Z}) = \Gamma_p/[\Gamma_p, \Gamma_p] = \mathbb{Z}^2 \oplus \mathbb{Z}_p.$$

Note that these Γ_p's produce infinitely many distinct nilmanifolds

$$\mathcal{N}_p = \mathcal{N}/\Gamma_p$$

covered by $\mathcal{N}_1$. We shall call

$$\mathcal{N}_1 = \mathcal{N}/\Gamma_1$$

the *standard nilmanifold.*

The classifying finite group actions on a 3-dimensional nilmanifold can be understood by the works of Bieberbach, L. Auslander and Waldhausen [6, 7, 14]. Free actions of cyclic, abelian and finite groups on the 3-torus were studied in [8], [11] and [5], respectively. If a finite group G acts freely on the standard nilmanifold $\mathcal{N}_1$, then either G is cyclic, or there does not exist any finite group acting freely on the standard nilmanifold $\mathcal{N}_1$ which yields an infra-nilmanifold homeomorphic to $\mathcal{N}/\pi_3$ or $\mathcal{N}/\pi_4$([3]). Free actions of finite abelian groups on the 3-dimensional nilmanifold $\mathcal{N}_p$ with the first homology $\mathbb{Z}^2 \oplus \mathbb{Z}_p$ were classified in [1]. Recently, the results of [1] were generalized without the abelian condition([2]).

We are interested in finding all free actions by finite groups G on the nilmanifold $\mathcal{N}_p$, under the condition that no translations of $\mathcal{N}$ are allowed, except for the central translations, which we shall call *a homotopically trivial translation.* That means we need to study $\widetilde{G}$ in

$$1 \longrightarrow \Gamma_p \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1.$$

If the G is finite and the action is free, then $\mathcal{N}_p/G$ is again a manifold whose fundamental group is $\widetilde{G}$. In fact, $\mathcal{N}_p/G = \mathcal{N}/\widetilde{G}$ is homeomorphic to an infra-nilmanifold. In other words, there is an imbedding of $\widetilde{G}$ into the affine group $\mathrm{Aff}(\mathcal{N}) = \mathcal{N} \rtimes \mathrm{Aut}(\mathcal{N})$. Such a group is called an *almost Bieberbach group.* Since all almost Bieberbach groups for $\mathcal{N}$ are classified already, all we need to do is, for each 3-dimensional almost Bieberbach group π, finding a normal subgroup N of π which is isomorphic to Γ_p. Then we describe the action of the finite quotient $G = \pi/N$ on the nilmanifold $\mathcal{N}_p = \mathcal{N}/\Gamma_p$. This G-action on $\mathcal{N}_p$ will be free.

Suppose there are two normal subgroups N_1, N_2 of π. The two actions of $\pi/N_1, \pi/N_2$ are *equivalent* if and only if there exists a homeomorphism f of $\mathcal{N}$ which conjugates the pair (N_1, π) into (N_2, π). Of course, such a conjugation is achieved by an affine map $f \in \mathrm{Aff}(\mathcal{N})$.

The following is the list for 15 kinds of the 3-dimensional almost Bieberbach groups imbedded in $\mathrm{Aff}(\mathcal{N}) = \mathcal{N} \rtimes (\mathbb{R}^2 \rtimes \mathrm{GL}(2,\mathbb{R}))$ ([2, p.1414]). We shall use

$$t_1 = \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I \right), \quad t_2 = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, I \right), \quad t_3 = \left(\begin{bmatrix} 1 & 0 & -\frac{1}{K} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I \right),$$

respectively, where I is the identity in $\mathrm{Aut}(\mathcal{N}) = \mathbb{R}^2 \rtimes \mathrm{GL}(2,\mathbb{R})$. In each presentation, n is any positive integer and t_3 is central except π_3 and π_4. Note that t_1 and t_2 are fixed, but K in t_3 varies for each $\pi_{i,j}$. For example, $K = n$ for π_1; $K = 2n$ for π_2, etc.

$\pi_1 = \langle\, t_1, t_2, t_3 \mid [t_2, t_1] = t_3^n \,\rangle,$

$\pi_2 = \langle\, t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{2n},\ \alpha^2 = t_3, \alpha t_1 \alpha^{-1} = t_1^{-1}, \alpha t_2 \alpha^{-1} = t_2^{-1} \,\rangle,$

$$\begin{aligned}\pi_3 = \langle\, t_1, t_2, t_3, \alpha \mid\ & [t_2, t_1] = t_3^{2n},\ [t_3, t_1] = [t_3, t_2] = 1,\ \alpha t_3 \alpha^{-1} = t_3^{-1}, \\ & \alpha t_1 \alpha^{-1} = t_1,\ \alpha t_2 = t_2^{-1} \alpha t_3^{-n},\ \ \alpha^2 = t_1 \,\rangle,\end{aligned}$$

$$\begin{aligned}\pi_4 = \langle\, t_1, t_2, t_3, \alpha, \beta \mid\ & [t_2, t_1] = t_3^{4n},\ [t_3, t_1] = [t_3, t_2] = [\alpha, t_3] = 1, \\ & \beta t_3 \beta^{-1} = t_3^{-1}, \alpha t_1 = t_1^{-1} \alpha t_3^{2n}, \alpha t_2 = t_2^{-1} \alpha t_3^{-2n}, \\ & \alpha^2 = t_3, \beta^2 = t_1, \beta t_1 \beta^{-1} = t_1, \beta t_2 = t_2^{-1} \beta t_3^{-2n}, \\ & \alpha\beta = t_1^{-1} t_2^{-1} \beta \alpha t_3^{-(2n+1)} \,\rangle,\end{aligned}$$

$\pi_{5,1} = \langle\, t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{4n-2},\ \alpha t_1 \alpha^{-1} = t_2, \alpha t_2 \alpha^{-1} = t_1^{-1}, \alpha^4 = t_3 \,\rangle,$

$\pi_{5,2} = \langle\, t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{4n}, \alpha t_1 \alpha^{-1} = t_2, \alpha t_2 \alpha^{-1} = t_1^{-1}, \alpha^4 = t_3^3 \,\rangle,$

$\pi_{5,3} = \langle\, t_1, t_2, t_3, \alpha |\, [t_2, t_1] = t_3^{4n}, \alpha t_1 \alpha^{-1} = t_2,\ \alpha t_2 \alpha^{-1} = t_1^{-1},\ \alpha^4 = t_3 \,\rangle,$
$\pi_{6,1} = \langle\, t_1, t_2, t_3, \alpha |\, [t_2, t_1] = t_3^{3n}, \alpha t_1 \alpha^{-1} = t_2, \alpha t_2 \alpha^{-1} = t_1^{-1} t_2^{-1}, \alpha^3 = t_3 \,\rangle,$
$\pi_{6,2} = \langle\, t_1, t_2, t_3, \alpha |\, [t_2, t_1] = t_3^{3n}, \alpha t_1 \alpha^{-1} = t_2, \alpha t_2 \alpha^{-1} = t_1^{-1} t_2^{-1}, \alpha^3 = t_3^2 \,\rangle,$
$\pi_{6,3} = \langle\, t_1, t_2, t_3, \alpha |\, [t_2, t_1] = t_3^{3n-2}, \alpha t_1 \alpha^{-1} = t_2, \alpha t_2 \alpha^{-1} = t_1^{-1} t_2^{-1}, \alpha^3 = t_3^2 \,\rangle,$
$\pi_{6,4} = \langle\, t_1, t_2, t_3, \alpha |\, [t_2, t_1] = t_3^{3n-1}, \alpha t_1 \alpha^{-1} = t_2, \alpha t_2 \alpha^{-1} = t_1^{-1} t_2^{-1}, \alpha^3 = t_3 \,\rangle,$
$\pi_{7,1} = \langle\, t_1, t_2, t_3, \alpha |\, [t_2, t_1] = t_3^{6n}, \alpha t_1 \alpha^{-1} = t_1 t_2, \alpha t_2 \alpha^{-1} = t_1^{-1}, \alpha^6 = t_3 \,\rangle,$
$\pi_{7,2} = \langle\, t_1, t_2, t_3, \alpha |\, [t_2, t_1] = t_3^{6n-2}, \alpha t_1 \alpha^{-1} = t_1 t_2, \alpha t_2 \alpha^{-1} = t_1^{-1}, \alpha^6 = t_3 \,\rangle,$
$\pi_{7,3} = \langle\, t_1, t_2, t_3, \alpha |\, [t_2, t_1] = t_3^{6n},\ \alpha t_1 \alpha^{-1} = t_1 t_2, \alpha t_2 \alpha^{-1} = t_1^{-1}, \alpha^6 = t_3^5 \,\rangle,$
$\pi_{7,4} = \langle\, t_1, t_2, t_3, \alpha |\, [t_2, t_1] = t_3^{6n-4},\ \alpha t_1 \alpha^{-1} = t_1 t_2, \alpha t_2 \alpha^{-1} = t_1^{-1}, \alpha^6 = t_3^5 \,\rangle.$

In this paper, we showed that if a finite group G acting freely on $\mathcal{N}_p$ with homotopically trivial translations, then either G is cyclic, or there exist finite nonabelian groups acting freely on $\mathcal{N}_p$ which yield orbit manifolds homeomorphic to $\mathcal{N}/\pi_3$ or $\mathcal{N}/\pi_4$. Note that our results cannot be obtained directly from [2], and differ in the following two respects. Firstly it is very hard to find a necessary and sufficient condition for being a normal nilpotent subgroup isomorphic to Γ_p of an almost Bieberbach group, because of many unknown variables. But a necessary and sufficient condition for being a normal subgroup can be obtained using a conjugation by the second Bieberbach theorem in the case of homotopically trivial translations. Second, since the finite groups acting freely on $\mathcal{N}_p$ in [2] are represented by generators, it is difficult to know the groups exactly. But in this paper, we show that there exist finite nonabelian group actions only in two classes π_3, π_4, and those are the dihedral groups D_k, D_{2k}, dicyclic groups $Dic_{\frac{k}{2}}$, Dic_k, or G_k in Theorem 3.1.

Let G be a finite group acting freely on the nilmanifold $\mathcal{N}_p$. Then clearly, $M = \mathcal{N}_p/G$ is a topological manifold, and $\pi = \pi_1(M) \subset \mathrm{Homeo}(\mathcal{N})$ is isomorphic to an almost Bieberbach group. Let π' be an embedding of π into $\mathrm{Aff}(\mathcal{N})$. Such an embedding always exists. Since any isomorphism between lattices extends uniquely to an automorphism of $\mathcal{N}$, we may assume the subgroup Γ_p goes to itself by the embedding $\pi \to \pi' \subset \mathrm{Aff}(\mathcal{N})$. From now on, we shall abuse the same notation Γ_p in $\mathrm{Aff}(\mathcal{N})$. Then the quotient group $G' = \pi'/\Gamma_p$ acts freely on the nilmanifold $\mathcal{N}_p = \mathcal{N}/\Gamma_p$. Moreover, $M' = \mathcal{N}_p/G'$ is an infra-nilmanifold. Thus, a finite free topological action $(G, \mathcal{N}_p)$ gives rise to an isometric action $(G', \mathcal{N}_p)$ on the nilmanifold $\mathcal{N}_p$. Clearly, $\mathcal{N}_p/G$ and $\mathcal{N}_p/G'$ are sufficiently large, see [7, Proposition 2]. By works of Waldhausen and Heil [6, 14], M is homeomorphic to M'.

Definition 1.1. Let groups G_i act on manifolds M_i, for $i = 1, 2$. The action (G_1, M_1) is *topologically conjugate* to (G_2, M_2) if there exists an isomorphism $\theta : G_1 \to G_2$ and a homeomorphism $h : M_1 \to M_2$ such that

$$h(g \cdot x) = \theta(g) \cdot h(x)$$

for all $x \in M_1$ and all $g \in G_1$. When $G_1 = G_2$ and $M_1 = M_2$, topologically conjugate is the same as *weakly equivariant*.

For $\mathcal{N}_p/G$ and $\mathcal{N}_p/G'$ being homeomorphic implies that the two actions $(G, \mathcal{N}_p)$ and $(G', \mathcal{N}_p)$ are topologically conjugate. Consequently, a finite free action $(G, \mathcal{N}_p)$ is topologically conjugate to an isometric action $(G', \mathcal{N}_p)$. Such a pair $(G', \mathcal{N}_p)$ is not unique. However, by the result obtained by Lee and Raymond [10], all the others are topologically conjugate.

Definition 1.2. Let $\pi \subset \mathrm{Aff}(\mathcal{N}) = \mathcal{N} \rtimes \mathrm{Aut}(\mathcal{N})$ be an almost Bieberbach group, and let N_1, N_2 be subgroups of π. We say that (N_1, π) is *affinely conjugate* to (N_2, π), denoted by $N_1 \sim N_2$, if there exists an element $(t, T) \in \mathrm{Aff}(\mathcal{N})$ such that $(t, T)\pi(t, T)^{-1} = \pi$ and $(t, T)N_1(t, T)^{-1} = N_2$.

Our classification problem of free finite group actions $(G, \mathcal{N}_p)$ with

$$\pi_1(\mathcal{N}_p/G) \cong \pi$$

can be solved by finding all normal nilpotent subgroups N of π each of which is isomorphic to Γ_p, and classify (N, π) up to affine conjugacy. This procedure is a purely group-theoretic problem and can be handled by affine conjugacy.

2. Criteria for affine conjugacy

In this section, we develop a technique for finding and classifying all possible finite group actions on the 3-dimensional nilmanifold. Let us define $\zeta(x) = \left(\begin{bmatrix} 1 & 0 & -x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I \right)$. Then $\Gamma_p = \langle t_1, t_2, \zeta(1/p) \rangle$ is a lattice of $\mathcal{N}$ such that $[t_2, t_1] = \zeta(1/p)^p = \zeta(1)$.

In this paper, we shall deal with the free action of a finite group G acting on $\mathcal{N}_p$ with homotopically trivial translations. Our situation is as follows. Let π be an almost Bieberbach group, and N' be its nil-radical (maximal normal nilpotent subgroup). Let N be a normal nilpotent subgroup of π. Suppose N'/N is generated by a central element of the nilpotent Lie group. Then we have a following diagram:

$$\begin{array}{ccccccccc}
 & & N & \xrightarrow{=} & N & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & N' & \longrightarrow & \pi & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{=} & & \\
1 & \longrightarrow & \mathbb{Z}_k & \longrightarrow & G & \longrightarrow & \Phi & \longrightarrow & 1
\end{array}$$

Note that $N \otimes \mathcal{Z} = N' \otimes \mathcal{Z}$, where $\mathcal{Z}$ is the center of the nilpotent Lie group $\mathcal{N}$. In other words, N and N' differ only in the central direction, which implies that the translation part of the action of G on the nilmanifold $\mathcal{N}/N$ is $\mathbb{Z}_k$. We will classify all such pairs (π, N)'s. For an almost Bieberbach group π, we want to find all normal subgroups N which satisfy

(a) N is isomorphic to the standard lattice Γ_p,
(b) $N'/(N' \cap \mathcal{Z}(\mathcal{N})) = N/(N \cap \mathcal{Z}(\mathcal{N}))$ in $\mathcal{N}/\mathcal{Z}(\mathcal{N})$.

The condition (b) comes from homotopically trivial translations. It is well known that, for any lattice Γ of $\mathcal{N}$, $\Gamma \cap (\mathcal{Z}(\mathcal{N}))$ is a lattice of $\mathcal{Z}(\mathcal{N})$. Thus such an Γ fits the short exact sequence

$$1 \longrightarrow \mathbb{Z} = \mathcal{Z}(\Gamma) \longrightarrow \Gamma \longrightarrow \mathbb{Z}^2 \longrightarrow 1.$$

Clearly $\mathbb{Z}^2$ is generated by the images of $\{t_1, t_2\}$ and $\mathbb{Z}$ is generated by $\zeta(1/K)$. Therefore there exists a generating set of Γ consisting of

$$\Gamma = \{t_1 {t_3}^u,\ t_2 {t_3}^v,\ [t_2, t_1]^{1/p} = \zeta(1/p)\}.$$

Note that there exists a conjugation which maps Γ_p onto Γ by the second Bieberbach theorem. In fact, let

$$J = t_1^{-\frac{v}{K}} t_2^{\frac{u}{K}},$$

and let μ_J denote the conjugation by J. Then clearly

$$\mu_J(t_1) = t_1 {t_3}^u,$$
$$\mu_J(t_2) = t_2 {t_3}^v.$$

We denote this image by $N(u, v)$. That is,

$$N(u, v) = \langle t_1 {t_3}^u, t_2 {t_3}^v,\ \zeta(1/p)\rangle$$

for $u, v \in \mathbb{Z}$. Therefore, $N(u,v)$ is the most general subgroup of $\mathcal{N}$ that satisfies (a) and (b) above.

Conditions on $N(u,v)$:

The condition $\zeta(1/p) \in \pi$ is necessary for $N(u,v)$ to be a subgroup of π. Since $\zeta(1/K)$ is one of the generators of π,

(1) p must divide K, say, $K = kp$.

Now $\mu_{J^{-1}}$ will map $N(u,v)$ onto Γ_p. That is,

$$\begin{aligned} \mu_{J^{-1}}(t_1 {t_3}^u) &= t_1, \\ \mu_{J^{-1}}(t_2 {t_3}^v) &= t_2, \\ \mu_{J^{-1}}(t_3) &= t_3. \end{aligned}$$

We also need $N(u,v)$ to be normal in π. Let $\alpha \in \pi$ be an element of a non-trivial holonomy. From now on, we shall use the notation $\widehat{\alpha} = \mu_{J^{-1}}(\alpha)$ and $\widehat{\pi} = \mu_{J^{-1}}(\pi)$. Then we have $\mu_\alpha(t_1 {t_3}^u),\ \mu_\alpha(t_2 {t_3}^v) \in N(u,v)$. This is equivalent to

$$\mu_{\widehat{\alpha}}(t_1),\ \mu_{\widehat{\alpha}}(t_2)\ \in \mu_{J^{-1}}(N(u,v)) = \Gamma_p.$$

When we write them as products of t_i's, we can get

$$\begin{aligned} \mu_{\widehat{\alpha}}(t_1) &= {t_1}^{n_1} {t_2}^{n_2} ({t_3}^k)^{n_3}, \\ \mu_{\widehat{\alpha}}(t_2) &= {t_1}^{m_1} {t_2}^{m_2} ({t_3}^k)^{m_3}. \end{aligned}$$

Since $n_i, m_i (i = 1, 2)$ are integers,

(2) Both n_3 and m_3 are integers.

Note that

$$N(u + ka, v + kb) = \langle (t_1 {t_3}^u)({t_3}^k)^a, (t_2 {t_3}^v)({t_3}^k)^b,\ \zeta(1/p) \rangle = N(u,v),$$

where u, v take integer values $0, 1, 2, \cdots, k-1$.

From the above two conditions (1) and (2), we can determine the form of a normal subgroup $N(u,v)$. Next we analyze when the pairs $\{u,v\}$ yield distinct $\widehat{\alpha}$'s. In order to denote $\widehat{\alpha}$ clearly, we rather write it as the following form

$$\widehat{\alpha} = T \cdot \alpha = \left({t_1}^{\ell_1} {t_2}^{\ell_2} {t_3}^{\ell_3} \right) \cdot \alpha$$

and look into $T \in \mathcal{N}$.

Finally we try to determine the finite group $G = \widehat{\pi}/\Gamma_p$. It is an extension of a cyclic group $\mathbb{Z}_k$ by the holonomy group Φ of π, where $\mathbb{Z}_k$ is the quotient $\frac{\mathcal{Z}(\mathcal{N}) \cap\ \widehat{\pi}}{\mathcal{Z}(\mathcal{N}) \cap \Gamma_p}$. Note that G fits the following extension

$$1 \longrightarrow \mathbb{Z}_k \longrightarrow G \longrightarrow \Phi \longrightarrow 1.$$

For each generator of the holonomy group Φ, we analyze the action. Let $\alpha = (a, A) \in \mathcal{N} \rtimes \mathrm{Aut}(\mathcal{N})$, and A have order d (holonomy order of α). Then we can write

$$\alpha^d = {t_1}^{d_1} {t_2}^{d_2} {t_3}^{d_3}.$$

In particular, we will show that if there exists an element α satisfying $d_3 \neq 0$, then G is cyclic of order $d(\frac{K}{p}) = dk$ which is generated by the image of $\widehat{\alpha}$ or $\widehat{\alpha}^{-1} t_3$. (see Theorem 3.1)

3. Free actions on $\mathcal{N}_p$ with orbit space $\mathcal{N}/\pi$

For each almost Bieberbach group π, we list all possible $N(u, v)$ and corresponding $\widehat{\alpha}$. In all cases, p must divide $K(= kp)$. Recall that $t_3 = [t_2, t_1]^{\frac{1}{K}}$ is a generator of $\widehat{\pi}$, and $[t_2, t_1]^{\frac{1}{p}} = \zeta(1/p) \in \Gamma_p$. Since

$$[t_2, t_1]^{\frac{1}{p}} = ([t_2, t_1]^{\frac{1}{K}})^{\frac{K}{p}} = (t_3)^{\frac{K}{p}} = {t_3}^k,$$

we have

$$\Gamma_p = \langle t_1, t_2, {t_3}^k \rangle$$

with $[t_2, t_1] = ({t_3}^k)^p$. We shall denote these standard generators for Γ_p by s_i such as

$$s_1 = t_1, s_2 = t_2, s_3 = {t_3}^k$$

so that $[s_2, s_1] = {s_3}^p$.

Let $N(u, v) = \langle t_1 {t_3}^u,\ t_2 {t_3}^v,\ {t_3}^k \rangle \cong \Gamma_p$ be a normal subgroup of π. Then the conjugation by J^{-1} maps

$$\begin{aligned} \mu_{J^{-1}}(t_1 {t_3}^u) &= t_1 = s_1, \\ \mu_{J^{-1}}(t_2 {t_3}^v) &= t_2 = s_2, \\ \mu_{J^{-1}}(t_3) &= t_3 = {s_3}^{\frac{1}{k}}. \end{aligned}$$

Therefore $\mu_{J^{-1}}$ maps $N(u, v)$ onto the standard Γ_p, and π to $\widehat{\pi}$. Thus $\langle t_1 {t_3}^u,\ t_2 {t_3}^v,\ {t_3}^k \rangle$ is normal in π if and only if $\Gamma_p = \langle s_1,\ s_2,\ s_3 \rangle$ is normal in $\widehat{\pi}$. Using this fact, we can classify all free actions on $\mathcal{N}_p$ with orbit space $\mathcal{N}/\pi$. This was done by the program Mathematica[15] and hand-checked.

THEOREM 3.1. *The groups that act on $\mathcal{N}_p$ freely with no translations except for homotopy-trivialities are described as follows:*

Table 1

G	Generator of G	$\mathcal{N}_p/G$	Conditions on u, v	Conditions on $K = kp$
$\mathbb{Z}_k$	$t_3 = {s_3}^{\frac{p}{K}}$	π_1	$u=0, v=0$	$n = kp$
$\mathbb{Z}_{2k}$	$\widehat{\alpha} = \alpha$	π_2	$u=0, v=0$	$2n = kp$
	$\widehat{\alpha} = {s_1}^{\frac{1}{p}} \cdot \alpha$		$u=0, v=\frac{k}{2}$	$k \in 2\mathbb{N}, p > 1, 2n = kp$
	$\widehat{\alpha} = \left({s_1}^{\frac{1}{p}} {s_2}^{-\frac{1}{p}} {s_3}^{-\frac{1}{2p}}\right) \cdot \alpha$		$u=\frac{k}{2}, v=\frac{k}{2}$	$k \in 2\mathbb{N}, p > 1, 2n = kp$
D_k	$t_3, \widehat{\alpha} = \alpha$	π_3	$u=0, v=0$	$p \in 2\mathbb{N}, 2n = kp$
$Dic_{\frac{k}{2}}$	$t_3, \widehat{\alpha} = \left({s_2}^{-\frac{1}{p}} {s_3}^{\frac{1}{4}}\right) \cdot \alpha$		$u=\frac{k}{2}, v=0$	$k \in 2\mathbb{N}, p \in 2\mathbb{N}, 2n = kp$
D_{2k}	$\widehat{\alpha} = \alpha$, $\widehat{\beta} = \beta$	π_4	$u=0, v=0$	$p \in 2\mathbb{N}, 4n = kp$
G_k	$\widehat{\alpha} = \left({s_1}^{\frac{1}{p}} {s_3}^{-\frac{1}{4}}\right) \cdot \alpha, \widehat{\beta} = \beta$		$u=0, v=\frac{k}{2}$	$k, p \in 2\mathbb{N}, 4n = kp$
Dic_k	$\widehat{\alpha} = \left({s_1}^{\frac{1}{p}} {s_2}^{-\frac{1}{p}} {s_3}^{-\frac{1}{2p}-\frac{1}{2}}\right) \cdot \alpha,$ $\widehat{\beta} = \left({s_2}^{-\frac{1}{p}} {s_3}^{\frac{1}{4}}\right) \cdot \beta$		$u=\frac{k}{2}, v=\frac{k}{2}$	$k, p \in 2\mathbb{N}, 4n = kp$
$\mathbb{Z}_{4k}$	$\widehat{\alpha} = \alpha$	$\pi_{5,1}$	$u=0, v=0$	$4n-2 = kp$
	$\widehat{\alpha} = \left({s_2}^{-\frac{1}{p}} {s_3}^{\frac{1}{4p}}\right) \cdot \alpha$		$u=\frac{k}{2}, v=\frac{k}{2}$	$k \in 2\mathbb{N}, 4n-2 = kp$
	$\widehat{\alpha}^{-1} t_3 = {s_3}^{\frac{p}{K}} \cdot \alpha^{-1}$	$\pi_{5,2}$	$u=0, v=0$	$4n = kp$
	$\widehat{\alpha}^{-1} t_3 = \left({s_1}^{\frac{1}{p}} {s_3}^{\frac{p}{K}-\frac{1}{4p}}\right) \cdot \alpha^{-1}$		$u=\frac{k}{2}, v=\frac{k}{2}$	$k \in 2\mathbb{N}, p > 1, 4n = kp$
	$\widehat{\alpha} = \alpha$	$\pi_{5,3}$	$u=0, v=0$	$4n = kp$
	$\widehat{\alpha} = \left({s_2}^{-\frac{1}{p}} {s_3}^{\frac{1}{4p}}\right) \cdot \alpha$		$u=\frac{k}{2}, v=\frac{k}{2}$	$k \in 2\mathbb{N}, p > 1, 4n = kp$
$\mathbb{Z}_{3k}$	$\widehat{\alpha} = \alpha$	$\pi_{6,1}$	$u=0, v=0$	$3n = kp$
	$\widehat{\alpha} = \left({s_2}^{-\frac{1}{p}} {s_3}^{-\frac{1}{6}+\frac{1}{6p}}\right) \cdot \alpha$		$u=\frac{k}{3}, v=\frac{k}{3}$	$k \in 3\mathbb{N}, p \geq 2, 3n = kp$
	$\widehat{\alpha} = \left({s_2}^{-\frac{2}{p}} {s_3}^{-\frac{1}{3}+\frac{2}{3p}}\right) \cdot \alpha$		$u=\frac{2k}{3}, v=\frac{2k}{3}$	$k \in 3\mathbb{N}, p \geq 3, 3n = kp$
	$\widehat{\alpha}^{-1} t_3 = {s_3}^{\frac{p}{K}} \cdot \alpha^{-1}$	$\pi_{6,2}$	$u=0, v=0$	$3n = kp$
	$\widehat{\alpha}^{-1} t_3 = \left({s_1}^{\frac{1}{p}} {s_3}^{\frac{p}{K}+\frac{1}{6}-\frac{1}{6p}}\right) \cdot \alpha^{-1}$		$u=\frac{k}{3}, v=\frac{k}{3}$	$k \in 3\mathbb{N}, p \geq 2, 3n = kp$
	$\widehat{\alpha}^{-1} t_3 = \left({s_1}^{\frac{2}{p}} {s_3}^{\frac{p}{K}+\frac{1}{3}-\frac{2}{3p}}\right) \cdot \alpha^{-1}$		$u=\frac{2k}{3}, v=\frac{2k}{3}$	$k \in 3\mathbb{N}, p \geq 3, 3n = kp$
	$\widehat{\alpha}^{-1} t_3 = {s_3}^{\frac{p}{K}} \cdot \alpha^{-1}$	$\pi_{6,3}$	$u=0, v=0$	$3n-2 = kp$
	$\widehat{\alpha} = \alpha$	$\pi_{6,4}$	$u=0, v=0$	$3n-1 = kp$
$\mathbb{Z}_{6k}$	$\widehat{\alpha} = \alpha$	$\pi_{7,1}$	$u=0, v=0$	$6n = kp$
	$\widehat{\alpha} = \alpha$	$\pi_{7,2}$	$u=0, v=0$	$6n-2 = kp$
	$\widehat{\alpha}^{-1} t_3 = {s_3}^{\frac{p}{K}} \cdot \alpha^{-1}$	$\pi_{7,3}$	$u=0, v=0$	$6n = kp$
	$\widehat{\alpha}^{-1} t_3 = {s_3}^{\frac{p}{K}} \cdot \alpha^{-1}$	$\pi_{7,4}$	$u=0, v=0$	$6n-4 = kp$

where $D_1 = \mathbb{Z}_2$, $D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$, $Dic_1 = \mathbb{Z}_4$, $G_2 = \mathbb{Z}_2 \times \mathbb{Z}_4$.

Proof. (Type 3.) We know that

$$\widehat{\pi_3} = \langle\, t_1, t_2, t_3, \widehat{\alpha} \,|\, [t_2, t_1] = t_3^{2n},\ [t_3, t_1] = [t_3, t_2] = 1,\ \widehat{\alpha} t_3 \widehat{\alpha}^{-1} = {t_3}^{-1}, \\ \widehat{\alpha} t_1 \widehat{\alpha}^{-1} = t_1 {t_3}^{-2u}, \widehat{\alpha} t_2 = t_2^{-1} \widehat{\alpha} {t_3}^{-n},\ \widehat{\alpha}^2 = t_1 {t_3}^{-u}\ \rangle,$$

$$\alpha = \left(\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) \right).$$

The family (π_3) is parametrized by $K = 2n$; K is divisible by p. Let $K = kp$. Conjugations by $\widehat{\alpha}$ yield

$$\mu_{\widehat{\alpha}}(s_1) = {s_1}^1 {s_2}^0 {s_3}^{-\frac{2pu}{K}},$$
$$\mu_{\widehat{\alpha}}(s_2) = {s_1}^0 {s_2}^{-1} {s_3}^{\frac{p}{2}},$$
$$\mu_{\widehat{\alpha}}(s_3) = {s_1}^0 {s_2}^0 {s_3}^{-1}.$$

The normal condition of Γ_p in $\widehat{\pi_3}$ requires that all the indices (superscripts) in the above be integers so that $-\frac{2u}{k},\ \frac{p}{2} \in \mathbb{Z}$. Therefore we assume p is even. Since $0 \le u < k$, we have $u = 0$ or $\frac{k}{2}$. Thus we have the following two types of normal nilpotent subgroups :

$$N(0, v) = \langle\, t_1, t_2 {t_3}^v, {t_3}^k \rangle, \quad N(k/2, v) = \langle\, t_1 {t_3}^{\frac{k}{2}}, t_2 {t_3}^v, {t_3}^k \rangle.$$

By using

$$\mu = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \left(\begin{bmatrix} \frac{v}{2n} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \right) \in N_{\mathrm{Aff}(\mathcal{N})}(\pi_3),$$

we can show that $N(0, v) \sim N(0, 0)$ and $N(k/2, v) \sim N(k/2, 0)$. Note that

$$\widehat{\alpha} = \mu_{J^{-1}}(\alpha) = \left({s_1}^0 {s_2}^{-\frac{2u}{K}} {s_3}^{\frac{pu}{2K}} \right) \cdot \alpha,$$
$$\widehat{\alpha}^2 = {s_1}^1 {s_2}^0 {s_3}^{-\frac{pu}{K}} = s_1 {s_3}^{-\frac{u}{k}}.$$

Hence we only need to deal with the following two cases:

(1) When $u = 0, v = 0$:

Since $G = \widehat{\pi_3}/\Gamma_p = \langle\, t_1, t_2, t_3, \widehat{\alpha} \rangle / \langle\, s_1, s_2, s_3 \rangle$, $\widehat{\alpha}^2 = s_1$ and $\widehat{\alpha} t_3 \widehat{\alpha}^{-1} = t_3^{-1}$, the finite group $G = \widehat{\pi_3}/\Gamma_p$ is represented by

$$G = \langle\, \bar{t_3}, \bar{\alpha} \mid \bar{t_3}^{\,k} = 1,\ \bar{\alpha}^2 = 1,\ \bar{\alpha} \bar{t_3} \bar{\alpha}^{-1} = \bar{t_3}^{\,-1}, p \in 2\mathbb{N}, k \in \mathbb{N}, kp = 2n \rangle,$$

which is isomorphic to the dihedral group D_k of order $2k$. Note that

$$G \text{ is abelian} \Leftrightarrow k = 1, p = 2n \text{ or } k = 2, p = n \Leftrightarrow G \text{ is } D_1 = \mathbb{Z}_2 \text{ or } D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2.$$

(2) When $u = \frac{k}{2}, v = 0$:

Recall that $G = \widehat{\pi_3}/\Gamma_p = \langle t_1, t_2, t_3, \widehat{\alpha} \rangle / \langle s_1, s_2, s_3 \rangle$. In this case, since $\widehat{\alpha}^2 = t_1 {t_3}^{-\frac{k}{2}} = s_1 {s_3}^{-\frac{1}{2}}$, we have $\bar{\alpha}^2 = \bar{t_3}^{\frac{k}{2}}$. So, we can induce that $G = \widehat{\pi_3}/\Gamma_p = \langle \bar{t_3}, \bar{\alpha} \rangle$ and G is represented by

$$G = \langle \bar{t_3}, \bar{\alpha} \mid \bar{t_3}^k = 1,\ \bar{\alpha}^2 = \bar{t_3}^{\frac{k}{2}},\ \bar{\alpha}\bar{t_3}\bar{\alpha}^{-1} = \bar{t_3}^{-1}, p \in 2\mathbb{N}, k \in 2\mathbb{N}, kp = 2n \rangle.$$

This group is isomorphic to the dicyclic group $Dic_{\frac{k}{2}}$ of order $2k$. Note that

$$G = \widehat{\pi_3}/\Gamma_p \text{ is abelian} \Leftrightarrow k = 2 \Leftrightarrow G = \langle \bar{\alpha} \rangle = Dic_1 = \mathbb{Z}_4,$$

where $\bar{\alpha}$ acts on $\mathcal{N}_p = \mathcal{N}/\Gamma_p$ by

$$\widehat{\alpha} = \left({s_1}^0 {s_2}^{-\frac{2u}{K}} {s_3}^{\frac{pu}{2K}} \right) \cdot \alpha.$$

Therefore we have the following five affinely non-conjugate actions:

$$D_1 = \mathbb{Z}_2, \quad D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2, \quad D_k (k \geq 3), \quad Dic_1 = \mathbb{Z}_4, \quad Dic_{\frac{k}{2}} (k \in 2\mathbb{N} + 2).$$

To summarize the above statements, the following table gives a complete list of all free actions of finite groups G on $\mathcal{N}_p$ which yield an orbit manifold homeomorphic to $\mathcal{N}/\pi_3$.

G	Conditions on u,v	Conditions on $K = kp$	Generator of G
$\mathbb{Z}_2$	$u = 0, v = 0$	$k = 1, 2n = p$	$\widehat{\alpha} = \alpha$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$u = 0, v = 0$	$k = 2, n = p \in 2\mathbb{N}$	t_3, $\widehat{\alpha} = \alpha$
D_k	$u = 0, v = 0$	$k \geq 3, p \in 2\mathbb{N}, 2n = kp$	t_3, $\widehat{\alpha} = \alpha$
$\mathbb{Z}_4$	$u = 1, v = 0$	$k = 2, p = n \in 2\mathbb{N}$	$\widehat{\alpha} = \left({s_2}^{-\frac{1}{p}} {s_3}^{\frac{1}{4}} \right) \cdot \alpha$
$Dic_{\frac{k}{2}}$	$u = \frac{k}{2}, v = 0$	$k \in 2\mathbb{N} + 2, p \in 2\mathbb{N}, 2n = kp$	t_3, $\widehat{\alpha} = \left({s_2}^{-\frac{1}{p}} {s_3}^{\frac{1}{4}} \right) \cdot \alpha$

(Type 4.) It is not hard to see that

$$\widehat{\pi_4} = \langle t_1, t_2, t_3, \widehat{\alpha}, \widehat{\beta} \mid [t_2, t_1] = t_3^{4n},\ [t_3, t_1] = [t_3, t_2] = [\widehat{\alpha}, t_3] = 1, \widehat{\beta} t_3 \widehat{\beta}^{-1} = t_3^{-1},$$

$$\widehat{\alpha} t_1 = t_1^{-1} \widehat{\alpha} {t_3}^{2n+2u},\ \widehat{\alpha} t_2 = t_2^{-1} \widehat{\alpha} {t_3}^{-2n+2v},\ \widehat{\alpha}^2 = t_3,\ \widehat{\beta}^2 = t_1 {t_3}^{-u},$$

$$\widehat{\beta} t_1 \beta^{-1} = t_1 {t_3}^{-2u}, \widehat{\beta} t_2 = t_2^{-1} \widehat{\beta} t_3^{-2n},\ \widehat{\alpha}\widehat{\beta} = t_1^{-1} t_2^{-1} \widehat{\beta} \widehat{\alpha} {t_3}^{-(2n+1)} {t_3}^{-(u+v)} \rangle,$$

$$\alpha = \left(\begin{bmatrix} 1 & 0 & -\frac{1}{8n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right) \right),$$
$$\beta = \left(\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{8} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) \right).$$

The family (π_4) is parametrized by $K = 4n$; K is divisible by p. Let $K = kp$. Conjugations by $\widehat{\alpha}$, $\widehat{\beta}$ yield

$$\mu_{\widehat{\alpha}}(s_1) = {s_1}^{-1}{s_2}^{0}{s_3}^{\frac{p(K+4u)}{2K}}, \qquad \mu_{\widehat{\beta}}(s_1) = {s_1}^{1}{s_2}^{0}{s_3}^{-\frac{2pu}{K}},$$
$$\mu_{\widehat{\alpha}}(s_2) = {s_1}^{0}{s_2}^{-1}{s_3}^{\frac{p(-K+4v)}{2K}}, \qquad \mu_{\widehat{\beta}}(s_2) = {s_1}^{0}{s_2}^{-1}{s_3}^{\frac{p}{2}},$$
$$\mu_{\widehat{\alpha}}(s_3) = {s_1}^{0}{s_2}^{0}{s_3}^{1}, \qquad \mu_{\widehat{\beta}}(s_3) = {s_1}^{0}{s_2}^{0}{s_3}^{-1}.$$

Since Γ_p is normal in $\widehat{\pi_4}$, we must have $\frac{2u}{k}$, $\frac{2v}{k}$, $\frac{p}{2} \in \mathbb{Z}$. Therefore, we assume p *is even.* Since $0 \le u, v < k$, we have $u, v = 0$ or $\frac{k}{2}$. Thus we have the following four types of normal nilpotent subgroups :

$$N_1 = N(0,0) = \langle t_1, t_2, {t_3}^k \rangle, \qquad N_2 = N(0,k/2) = \langle t_1, t_2{t_3}^{\frac{k}{2}}, {t_3}^k \rangle,$$
$$N_3 = N(k/2,0) = \langle t_1{t_3}^{\frac{k}{2}}, t_2, {t_3}^k \rangle, \quad N_4 = N(k/2,k/2) = \langle t_1{t_3}^{\frac{k}{2}}, t_2{t_3}^{\frac{k}{2}}, {t_3}^k \rangle.$$

It needs some calculations to obtain that the normalizer $N_{\mathrm{Aff}(\mathcal{N})}(\pi_4)$ is of the form

$$\mu = \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right),$$

where $2x, 2y \in \mathbb{Z}$, $z \in \mathbb{R}$, and $\left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)$ is one of the following eight values

$$\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right), \left(\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right), \left(\begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right), \left(\begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right),$$
$$\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right), \left(\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right), \left(\begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right), \left(\begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right).$$

By using

$$\mu = \left(\begin{bmatrix} 1 & 0 & \frac{1}{4K} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \right) \in N_{\mathrm{Aff}(\mathcal{N})}(\pi_4),$$

we can show that $N(0,k/2) \sim N(k/2,0)$.

Next, assume that N_1 is affinely conjugate to N_2. Then there exists an element

$$\mu_1 = \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right) \in N_{\mathrm{Aff}(\mathcal{N})}(\pi_4)$$

which can conjugate N_1 onto N_2. From this we must have $x = -u \pm \frac{1}{2p}$. However, since $u = 0, -\frac{1}{2}$ and $p \in 2\mathbb{N}$, $2x = -2u \pm \frac{1}{p}$ is not an integer. This is a contradiction. Thus N_1 is not affinely conjugate to N_2. Similarly, we can show that $N_1 \nsim N_4$ and $N_2 \nsim N_4$.

Note that

$$\begin{aligned}
\widehat{\alpha} &= \left(s_1{}^{\frac{2v}{K}} s_2{}^{-\frac{2u}{K}} s_3{}^{-\frac{p(4uv+K(u+v))}{2K^2}}\right) \cdot \alpha,\\
\widehat{\alpha}^2 &= s_1{}^0 s_2{}^0 s_3{}^{\frac{p}{K}} = s_3{}^{\frac{1}{k}} = t_3,\\
\widehat{\beta} &= \left(s_1{}^0 s_2{}^{-\frac{2u}{K}} s_3{}^{\frac{pu}{2K}}\right) \cdot \beta,\\
\widehat{\beta}^2 &= s_1{}^1 s_2{}^0 s_3{}^{-\frac{pu}{K}} = s_1 s_3{}^{-\frac{u}{k}}.
\end{aligned}$$

Hence, we only need to deal with the following three cases:

(1) When $u = 0, v = 0$:

Since $\widehat{\alpha}^2 = t_3 = s_3{}^{\frac{1}{k}}$, $\widehat{\beta}^2 = s_1$, $\widehat{\alpha} t_3 \widehat{\alpha}^{-1} = t_3$, and $\widehat{\alpha}\widehat{\beta} = t_1{}^{-1} t_2{}^{-1} \widehat{\beta}\widehat{\alpha} t_3{}^{-(2n+1)}$, we have $\widehat{\alpha}\widehat{\beta} = t_1{}^{-1} t_2{}^{-1} t_3{}^{(2n+1)} \widehat{\beta}\widehat{\alpha}$. So, by using $kp = 4n$ and $p \in 2\mathbb{N}$, we can obtain that

$$\bar{\alpha}\bar{\beta} = \bar{t_3}\bar{\beta}\bar{\alpha} \quad \Leftrightarrow \quad \bar{\beta}\bar{\alpha}\bar{\beta}^{-1} = \bar{\alpha}^{-1}.$$

Therefore the finite group $G = \widehat{\pi_4}/\Gamma_p = \langle\, t_1, t_2, t_3, \widehat{\alpha}, \widehat{\beta}\rangle / \langle\, s_1, s_2, s_3\rangle$ is represented by

$$G = \langle\, \bar{\alpha},\, \bar{\beta} \mid \bar{\alpha}^{2k} = 1,\, \bar{\beta}^2 = 1,\, \bar{\beta}\bar{\alpha}\bar{\beta}^{-1} = \bar{\alpha}^{-1}, p \in 2\mathbb{N}, k \in \mathbb{N}, kp = 4n\rangle,$$

which is isomorphic to the dihedral group D_{2k} of order $4k$. Note that

$$G \text{ is abelian} \Leftrightarrow k = 1, p = 4n \Leftrightarrow G \text{ is } D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2.$$

(2) When $u = 0, v = \frac{k}{2}$:

In this case, since $\widehat{\alpha}^2 = t_3 = s_3{}^{\frac{1}{k}}$, $\widehat{\beta}^2 = s_1$, $\widehat{\alpha} t_3 \widehat{\alpha}^{-1} = t_3$, $kp = 4n$, and $p \in 2\mathbb{N}$, using the following relations,

$$\widehat{\alpha}\widehat{\beta} = t_1^{-1} t_2^{-1} \widehat{\beta}\widehat{\alpha} t_3^{-(2n+1)} t_3{}^{-\frac{k}{2}} = t_1^{-1} t_2^{-1} t_3{}^{(2n+1)} t_3{}^{\frac{k}{2}} \widehat{\beta}\widehat{\alpha} = t_1^{-1} t_2^{-1} t_3{}^{2n} \widehat{\alpha}^2 \widehat{\alpha}^k \widehat{\beta}\widehat{\alpha},$$

we can induce that $\bar{\alpha}\bar{\beta} = \bar{\alpha}^{k+2}\bar{\beta}\bar{\alpha} \Leftrightarrow \bar{\alpha} = \bar{\alpha}^{k+2}\bar{\beta}\bar{\alpha}\bar{\beta} \Leftrightarrow \bar{\beta}\bar{\alpha}\bar{\beta} = \bar{\alpha}^{-k-1} = \bar{\alpha}^{k-1}$.

Therefore the finite group $G = \widehat{\pi_4}/\Gamma_p = \langle\, t_1, t_2, t_3, \widehat{\alpha}, \widehat{\beta}\rangle / \langle\, s_1, s_2, s_3\rangle$ is represented by

$$G_k := \widehat{\pi_4}/\Gamma_p = \langle\, \bar{\alpha}, \bar{\beta} \mid \bar{\alpha}^{2k} = 1,\, \bar{\beta}^2 = 1,\, \bar{\beta}\bar{\alpha}\bar{\beta} = \bar{\alpha}^{k-1},\, p, k \in 2\mathbb{N}\rangle.$$

In particular, if $k = 2^{m-2}$, then G_k is isomorphic to the semidihedral group SD_{2^m} of order 2^m. Note that G_k is abelian $\Leftrightarrow k = 2 \Leftrightarrow G_2 = \mathbb{Z}_2 \times \mathbb{Z}_4$.

(3) When $u = \frac{k}{2}, v = \frac{k}{2}$:

Since $\widehat{\alpha}^2 = t_3 = s_3{}^{\frac{1}{k}}$, $\widehat{\beta}^2 = t_1 t_3{}^{-\frac{k}{2}}$, $\widehat{\alpha} t_3 \widehat{\alpha}^{-1} = t_3$, $kp = 4n$, and $p \in 2\mathbb{N}$, from the following relations,

$$\widehat{\alpha}\widehat{\beta} = t_1^{-1} t_2^{-1} \widehat{\beta}\widehat{\alpha} t_3^{-(2n+1)} t_3{}^{-k} = t_1^{-1} t_2^{-1} t_3{}^{2n} t_3{}^{k} \widehat{\beta}\widehat{\alpha} t_3{}^{-1} = t_1^{-1} t_2^{-1} t_3{}^{2n} t_3{}^{k} \widehat{\beta}\widehat{\alpha}^{-1},$$

we obtain that $\bar{\alpha}\bar{\beta} = \bar{\beta}\bar{\alpha}^{-1} \Leftrightarrow \bar{\alpha}\bar{\beta}\bar{\alpha} = \bar{\beta} \Leftrightarrow \bar{\beta}\bar{\alpha}\bar{\beta}^{-1} = \bar{\alpha}^{-1}$.

Therefore the finite group $G = \widehat{\pi_4}/\Gamma_p = \langle\, t_1, t_2, t_3, \widehat{\alpha}, \widehat{\beta}\rangle / \langle\, s_1, s_2, s_3\rangle$ is represented by

$$G = \widehat{\pi_4}/\Gamma_p = \langle\, \bar{\alpha}, \bar{\beta} \mid \bar{\alpha}^{2k} = 1,\, \bar{\beta}^2 = \bar{\alpha}^k,\, \bar{\beta}\bar{\alpha}\bar{\beta}^{-1} = \bar{\alpha}^{-1},\, p, k \in 2\mathbb{N}, kp = 4n\rangle.$$

This group is isomorphic to the dicyclic group Dic_k of order $4k$. Since $k \in 2\mathbb{N}$, $G = Dic_k$ is nonabelian. The generators $\bar{\alpha}$ and $\bar{\beta}$ act on $\mathcal{N}_p = \mathcal{N}/\Gamma_p$ by

$$\widehat{\alpha} = \left(s_1{}^{\frac{2v}{K}} s_2{}^{-\frac{2u}{K}} s_3{}^{-\frac{p(4uv+K(u+v))}{2K^2}}\right) \cdot \alpha, \quad \widehat{\beta} = \left(s_1{}^0 s_2{}^{-\frac{2u}{K}} s_3{}^{\frac{pu}{2K}}\right) \cdot \beta.$$

(Type 5.) Note that

$$\widehat{\pi_{5,2}} = \langle\, t_1, t_2, t_3, \widehat{\alpha} \,|\, [t_2, t_1] = t_3^{4n},\ \widehat{\alpha}^4 = t_3{}^3,\ \widehat{\alpha} t_1 \widehat{\alpha}^{-1} = t_2 t_3{}^{u-v}, \\ \widehat{\alpha} t_2 \widehat{\alpha}^{-1} = t_1^{-1} t_3{}^{u+v} \,\rangle,$$

$$\alpha = \left(\begin{bmatrix} 1 & 0 & -\frac{3}{16n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) \right).$$

The family $(\pi_{5,2})$ is parametrized by $K = 4n$; K is divisible by p. Let $K = kp$. Conjugations by $\widehat{\alpha}$ yield

$$\begin{aligned} \mu_{\widehat{\alpha}}(s_1) &= s_1{}^0 s_2{}^1 s_3{}^{\frac{p(u-v)}{K}}, \\ \mu_{\widehat{\alpha}}(s_2) &= s_1{}^{-1} s_2{}^0 s_3{}^{\frac{p(u+v)}{K}}, \\ \mu_{\widehat{\alpha}}(s_3) &= s_1{}^0 s_2{}^0 s_3{}^1. \end{aligned}$$

By the normality of Γ_p in $\widehat{\pi_{5,2}}$, we must have $\frac{p(u-v)}{K}$, $\frac{p(u+v)}{K} \in \mathbb{Z}$. Since $0 \le u, v < k$, we have $u(=v) = 0$ or $\frac{k}{2}$. Thus we have the following two normal nilpotent subgroups:

$$N_1 = N(0,0) = \langle\, t_1, t_2, t_3{}^k \rangle, \qquad N_4 = N(k/2, k/2) = \langle\, t_1 t_3{}^{\frac{k}{2}}, t_2 t_3{}^{\frac{k}{2}}, t_3{}^k \rangle.$$

It is not hard to seee that the normalizer $N_{\mathrm{Aff}(\mathcal{N})}(\pi_{5,i})$ is of the form

$$\mu = \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right),$$

where $x + y, x - y \in \mathbb{Z}$, $z \in \mathbb{R}$, and x^2 must be a multiple of $\frac{1}{K}$, and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \\ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

If $p = 1$, then it is easy to show that $N_1 \sim N_4$ by using

$$\mu = \left(\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \right) \in N_{\mathrm{Aff}(\mathcal{N})}(\pi_{5,2}).$$

Let $p>1$. If there exists an element

$$\mu = \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right) \in N_{\mathrm{Aff}(\mathcal{N})}(\pi_{5,2})$$

which can conjugate N_1 onto N_4, then we obtain that $x=\pm\frac{1}{2p}$ and $y=\pm\frac{1}{2p}$. So, $x+y\notin\mathbb{Z}$ or $x-y\notin\mathbb{Z}$. This is a contraction. Therefore N_1 is not affinely conjugate to N_4.

Note that

$$\widehat{\alpha} = \left({s_1}^{\frac{v-u}{K}} {s_2}^{-\frac{u+v}{K}} {s_3}^{\frac{pu^2}{K^2}} \right) \cdot \alpha,$$
$$\widehat{\alpha}^4 = {s_1}^0 {s_2}^0 {s_3}^{\frac{3p}{K}} = {s_3}^{\frac{3}{k}}.$$

Also, since $\widehat{\alpha} = (\widehat{\alpha}^{-1}t_3)^3$ and $t_3 = (\widehat{\alpha}^{-1}t_3)^4$, for any u, v, we have $(\widehat{\alpha}^{-1}t_3)^{4k} = s_3 \in \Gamma_p$. Hence,

$$G = \widehat{\pi_{5,2}}/\Gamma_p = \mathbb{Z}_{4k} = \langle \bar{\alpha}^{-1}\bar{t_3} \mid (\bar{\alpha}^{-1}\bar{t_3})^{4k} = 1 \rangle,$$

where $\bar{\alpha}^{-1}\bar{t_3}$ acts on $\mathcal{N}_p = \mathcal{N}/\Gamma_p$ by

$$\widehat{\alpha}^{-1}t_3 = \left({s_1}^{\frac{u+v}{K}} {s_2}^{\frac{v-u}{K}} {s_3}^{\frac{p}{K}-\frac{pu^2}{K^2}} \right) \cdot \alpha^{-1},$$

for $(u,v)=(0,0)$, $(k/2,k/2)$.

(Type 6.) Some calculations show that
$\widehat{\pi_{6,1}} = \langle\, t_1, t_2, t_3, \widehat{\alpha} \,|\, [t_2,t_1] = t_3^{3n},\ \widehat{\alpha}^3 = t_3,\ \widehat{\alpha}t_1\widehat{\alpha}^{-1} = t_2{t_3}^{u-v},\ \widehat{\alpha}t_2\widehat{\alpha}^{-1} = t_1^{-1}t_2^{-1}{t_3}^{u+2v} \,\rangle$,

$$\alpha = \left(\begin{bmatrix} 1 & 0 & -\frac{1}{9n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \right) \right).$$

The family $(\pi_{6,1})$ is parametrized by $K=3n$; K is divisible by p. Let $K=kp$. Conjugations by $\widehat{\alpha}$ yield

$$\mu_{\widehat{\alpha}}(s_1) = {s_1}^0 {s_2}^1 {s_3}^{\frac{p(u-v)}{K}},$$
$$\mu_{\widehat{\alpha}}(s_2) = {s_1}^{-1} {s_2}^{-1} {s_3}^{\frac{p(u+2v)}{K}},$$
$$\mu_{\widehat{\alpha}}(s_3) = {s_1}^0 {s_2}^0 {s_3}^1.$$

By the normality of Γ_p in $\widehat{\pi_{6,1}}$, we have $\frac{p(u-v)}{K}$, $\frac{p(u+2v)}{K} \in \mathbb{Z}$. Since $0 \le u, v < k$, we can conclude that $u (= v) = 0$, $\frac{k}{3}$, or $\frac{2k}{3}$. Thus we have the following three types of normal nilpotent subgroups :

$$N_1 = N(0,0) = \langle\, t_1, t_2, {t_3}^k \rangle,$$
$$N_2 = N(k/3,k/3) = \langle\, t_1{t_3}^{\frac{k}{3}}, t_2{t_3}^{\frac{k}{3}}, {t_3}^k \rangle,$$
$$N_3 = N(2k/3,2k/3) = \langle\, t_1{t_3}^{\frac{2k}{3}}, t_2{t_3}^{\frac{2k}{3}}, {t_3}^k \rangle.$$

By calculation, we obtain that the normalizer $N_{\mathrm{Aff}(\mathcal{N})}(\pi_{6,i})$ is of the form

$$\mu = \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right),$$

where $z \in \mathbb{R}$, and if $ad - bc = 1$, then $x + y \in \mathbb{Z}$, $-x + 2y \in \mathbb{Z}$, and

$$\left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \left(\begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right), \left(\begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \right), \left(\begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{6} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \right),$$
$$\left(\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \right), \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right), \left(\begin{bmatrix} \frac{1}{6} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \right);$$

and if $ad - bc = -1$, then $x + y \in \mathbb{Z}$, $2x - y \in \mathbb{Z}$, and

$$\left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \left(\begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right), \left(\begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \right), \left(\begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{6} \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \right),$$
$$\left(\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \right), \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right), \left(\begin{bmatrix} \frac{1}{6} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \right).$$

By using $\mu = \left(\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \right) \in N_{\mathrm{Aff}(\mathcal{N})}(\pi_{6,i})$, we can show that if $p = 1$, then $N_1 \sim N_2 \sim N_3$ and if $p = 2$, then $N_1 \sim N_3$. Let $p \geq 2$. In this case, we will show that N_1 is not affinely conjugate to N_2. Assume that if there exists

$$\mu = \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right) \in N_{\mathrm{Aff}(\mathcal{N})}(\pi_{6,i})$$

which can conjugate N_1 onto N_2, then μ is one of the following two types:

(1) when $ad - bc = 1$,

$$\left(\begin{bmatrix} 1 & -\frac{1}{3p} & z \\ 0 & 1 & \frac{1}{3p} \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \right), \left(\begin{bmatrix} 1 & \frac{-1+p}{3p} & z \\ 0 & 1 & \frac{1-p}{3p} \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right) \right).$$

(2) when $ad - bc = -1$,

$$\left(\begin{bmatrix} 1 & \frac{-1+p}{3p} & z \\ 0 & 1 & \frac{1-p}{3p} \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \right), \left(\begin{bmatrix} 1 & -\frac{1}{3p} & z \\ 0 & 1 & \frac{1}{3p} \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right) \right).$$

However, since $p \geq 2$, if $ad - bc = 1$, then $-x + 2y = \frac{1}{p}, \frac{1}{p} - 1 \notin \mathbb{Z}$, and if $ad - bc = -1$, then $2x - y = -\frac{1}{p} + 1, -\frac{1}{p} \notin \mathbb{Z}$. This is a contradiction. Therefore there does not exist $\mu \in N_{\mathrm{Aff}(\mathcal{N})}(\pi_{6,i})$ which conjugates N_1 onto N_2.

Similarly we can prove that if $p \geq 3$, then N_1 is not affinely conjugate to N_3, and if $p \geq 2$, then N_2 is not affinely conjugate to N_3. So, we can obtain that

$$\begin{aligned} p = 1 &\Longrightarrow N_1 \sim N_2 \sim N_3, \\ p = 2 &\Longrightarrow N_1 \sim N_3,\ N_1 \nsim N_2, \\ p \geq 3 &\Longrightarrow N_1 \nsim N_2,\ N_1 \nsim N_3,\ N_2 \nsim N_3. \end{aligned}$$

Note that

$$\begin{aligned} \widehat{\alpha} &= \left(s_1{}^{\frac{v-u}{K}} s_2{}^{-\frac{2u+v}{K}} s_3{}^{\frac{p(-Ku+3u^2)}{2K^2}}\right) \cdot \alpha, \\ \widehat{\alpha}^3 &= s_1{}^0 s_2{}^0 s_3{}^{\frac{p}{K}} = s_3{}^{\frac{1}{k}}. \end{aligned}$$

For any $u, v \in \mathbb{Z}$, we have $(\widehat{\alpha}^3)^k = s_3 \in \Gamma_p$. Therefore we can get

$$G = \widehat{\pi_{6,1}}/\Gamma_p = \mathbb{Z}_{3k} = \langle \bar{\alpha} \mid \bar{\alpha}^{3k} = 1 \rangle,$$

where $\bar{\alpha}$ acts on $\mathcal{N}_p = \mathcal{N}/\Gamma_p$ by

$$\widehat{\alpha} = \left(s_1{}^{\frac{v-u}{K}} s_2{}^{-\frac{2u+v}{K}} s_3{}^{\frac{p(-Ku+3u^2)}{2K^2}}\right) \cdot \alpha,$$

for $(u, v) = (0, 0)$, $(k/3, k/3)$, $(2k/3, 2k/3)$.

Next we deal with the case of

$$\begin{aligned} \widehat{\pi_{6,3}} = \langle\, t_1, t_2, t_3, \widehat{\alpha} \,|\, [t_2, t_1] = t_3^{3n-2},\ \widehat{\alpha}^3 = t_3^2,\ \widehat{\alpha} t_1 \widehat{\alpha}^{-1} = t_2 t_3{}^{u-v}, \\ \widehat{\alpha} t_2 \widehat{\alpha}^{-1} = t_1^{-1} t_2^{-1} t_3{}^{u+2v} \,\rangle, \end{aligned}$$

$$\alpha = \left(\begin{bmatrix} 1 & 0 & -\frac{2}{9n-6} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \left(\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \right) \right).$$

The family $(\pi_{6,3})$ is parametrized by $K = 3n - 2$; K is divisible by p. Let $K = kp$. Conjugations by $\widehat{\alpha}$ yield

$$\begin{aligned} \mu_{\widehat{\alpha}}(s_1) &= s_1{}^0 s_2{}^1 s_3{}^{\frac{p(u-v)}{K}}, \\ \mu_{\widehat{\alpha}}(s_2) &= s_1{}^{-1} s_2{}^{-1} s_3{}^{\frac{p(u+2v)}{K}}, \\ \mu_{\widehat{\alpha}}(s_3) &= s_1{}^0 s_2{}^0 s_3{}^1. \end{aligned}$$

Since Γ_p is normal in $\widehat{\pi_{6,3}}$, we must have $\frac{p(u-v)}{K}$, $\frac{p(u+2v)}{K} \in \mathbb{Z}$. Since $0 \leq u, v < k$, we can conclude that $u (= v) = 0$, $\frac{k}{3}$, or $\frac{2k}{3}$. In this case, since $K = kp = 3n - 2$, $\frac{k}{3} = \frac{3n-2}{3p}$ and $\frac{2k}{3} = \frac{2(3n-2)}{3p}$ cannot be integers. Thus we have only one normal nilpotent subgroup

$$N(0, 0) = \langle\, t_1, t_2, t_3{}^k \rangle.$$

Note that

$$\begin{aligned} \widehat{\alpha} &= \left(s_1{}^{\frac{v-u}{K}} s_2{}^{-\frac{2u+v}{K}} s_3{}^{\frac{p(-Ku+3u^2)}{2K^2}}\right) \cdot \alpha, \\ \widehat{\alpha}^3 &= s_1{}^0 s_2{}^0 s_3{}^{\frac{2p}{K}} = s_3{}^{\frac{2}{k}}. \end{aligned}$$

Also, since $\widehat{\alpha} = (\widehat{\alpha}^{-1}t_3)^2$ and $t_3 = (\widehat{\alpha}^{-1}t_3)^3$ for any u, v, we have $(\widehat{\alpha}^{-1}t_3)^{3k} = s_3 \in \Gamma_p$. Hence we obtain

$$G = \widehat{\pi_{6,3}}/\Gamma_p = \mathbb{Z}_{3k} = \langle \bar{\alpha}^{-1}\bar{t_3} \mid (\bar{\alpha}^{-1}\bar{t_3})^{3k} = 1 \rangle,$$

where $\bar{\alpha}^{-1}\bar{t_3}$ acts on $\mathcal{N}_p = \mathcal{N}/\Gamma_p$ by

$$\widehat{\alpha}^{-1}t_3 = s_3{}^{\frac{p}{K}} \cdot \alpha^{-1}.$$

The other cases can be done similarly. □

According to the Theorem 3.1, if $p = 1$, then we can obtain the following result which is the same as the Theorem 3.3 of [3].

Corollary 3.2. *Suppose G is a finite group acting freely on the standard nilmanifold $\mathcal{N}_1$ with no translations except for homotopy-trivialities. Then G is cyclic, and it is one of the following.*

Table 2

G	Generator of G	$\mathcal{N}_1/G$	Conditions on u, v	Conditions on $K = kp$
$\mathbb{Z}_k$	$t_3 = s_3{}^{\frac{1}{K}}$	π_1	$u = 0, v = 0$	$n = k = K$
$\mathbb{Z}_{2k}$	$\widehat{\alpha} = \alpha$	π_2	$u = 0, v = 0$	$2n = k = K$
$\mathbb{Z}_{4k}$	$\widehat{\alpha} = \alpha$	$\pi_{5,1}$	$u = 0, v = 0$	$4n - 2 = k = K$
	$\widehat{\alpha} = (s_2{}^{-1}s_3{}^{\frac{1}{4}}) \cdot \alpha$		$u = \frac{k}{2}, v = \frac{k}{2}$	$4n - 2 = k = K$
	$\widehat{\alpha}^{-1}t_3 = s_3{}^{\frac{1}{K}} \cdot \alpha^{-1}$	$\pi_{5,2}$	$u = 0, v = 0$	$4n = k = K$
	$\widehat{\alpha} = \alpha$	$\pi_{5,3}$	$u = 0, v = 0$	$4n = k = K$
$\mathbb{Z}_{3k}$	$\widehat{\alpha} = \alpha$	$\pi_{6,1}$	$u = 0, v = 0$	$3n = k = K$
	$\widehat{\alpha}^{-1}t_3 = s_3{}^{\frac{1}{K}} \cdot \alpha^{-1}$	$\pi_{6,2}$	$u = 0, v = 0$	$3n = k = K$
	$\widehat{\alpha}^{-1}t_3 = s_3{}^{\frac{1}{K}} \cdot \alpha^{-1}$	$\pi_{6,3}$	$u = 0, v = 0$	$3n - 2 = k = K$
	$\widehat{\alpha} = \alpha$	$\pi_{6,4}$	$u = 0, v = 0$	$3n - 1 = k = K$
$\mathbb{Z}_{6k}$	$\widehat{\alpha} = \alpha$	$\pi_{7,1}$	$u = 0, v = 0$	$6n = k = K$
	$\widehat{\alpha} = \alpha$	$\pi_{7,2}$	$u = 0, v = 0$	$6n - 2 = k = K$
	$\widehat{\alpha}^{-1}t_3 = s_3{}^{\frac{1}{K}} \cdot \alpha^{-1}$	$\pi_{7,3}$	$u = 0, v = 0$	$6n = k = K$
	$\widehat{\alpha}^{-1}t_3 = s_3{}^{\frac{1}{K}} \cdot \alpha^{-1}$	$\pi_{7,4}$	$u = 0, v = 0$	$6n - 4 = k = K$

In [1, 3], any finite group acting freely on the nilmanifold $\mathcal{N}_p$ is abelian. However, as we can see in the following example, if a finite group acts freely on $\mathcal{N}_p$ with homotopically trivial translations, there exist nonabelian groups which yield orbit manifolds homeomorphic to $\mathcal{N}/\pi_3$ or $\mathcal{N}/\pi_4$.

EXAMPLE 3.3. Let G be a finite group of order 16 acting freely on $\mathcal{N}_p(p \in 2\mathbb{N})$ with homotopically trivial translations. Then G is one of the following four groups:

$$\mathbb{Z}_{16},\ \text{dihedral group } D_8,\ \text{dicyclic group } Dic_4,\\ \text{semidihedral group } SD_{16} = G_4.$$

In each case, non-affinely conjugate actions are as follows:

- $\mathbb{Z}_{16}$: one in π_1, three in π_2, two in $\pi_{5,i}(i = 2, 3)$.
- D_8: one in $\pi_3(k = 8)$, one in $\pi_4(k = 4)$.
- Dic_4: one in $\pi_3(k = 8)$, one in $\pi_4(k = 4)$.
- SD_{16}: one in π_4.

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